

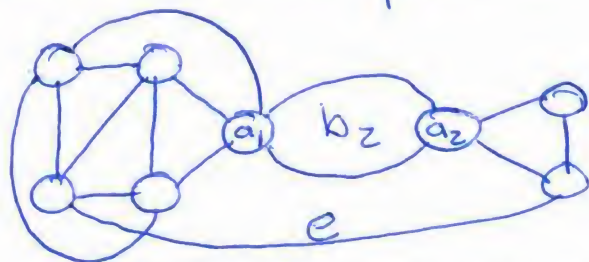
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Graph Theory Quiz 5 (21 June 2019)  
Open book, open notes, open neighbor.

1. For each of the following, prove that  $G$  must be planar or give a counter-example.

- (a)  $G$  is defined by  $G = (G' + e)$ , where  $G'$  can be defined by the block cut-point graph with blocks  $\{b_1, b_2, b_3\}$  and articulation vertices  $\{a_1, a_2\}$ .  $a_1$  connects blocks  $b_1$  and  $b_2$ ,  $a_2$  connects blocks  $b_2$  and  $b_3$ , and the edge  $e$  connects some vertex in  $b_1$  to some vertex in  $b_3$ .  $b_1$ ,  $b_2$ , and  $b_3$  are all planar.

Counter-example:  $e$  completes a Kuratowski subgraph

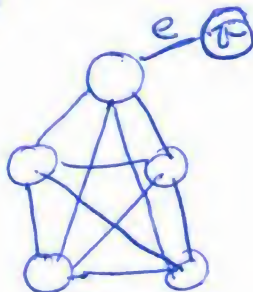


- (b)  $G$  is defined as  $G = (G' - e)$ , where  $G'$  is a minimal non-planar graph and  $e = (u, v)$  is some edge  $e \in E(G')$  such that  $d(u) = d(v) = \Delta(G)$ .

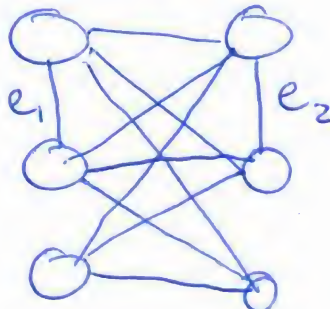
$G$  will always be planar by the definition of "minimal non-planar" for  $G'$

- (c)  $G$  has  $|V(G)| = 6$  and  $|E(G)| = 11$ .

$K_5 + v + e$



$K_{3,3} + e_1 + e_2$



2. Consider simple but possibly disconnected planar graph  $G$  where  $|V(G)| \geq 2$ . Prove that  $\exists u, v \in V(G) : d(u) < 5, d(v) < 5$ .

Consider the degree sum formula

$$\sum_{i \in V(G)} d_i = 2m$$

and a necessary condition for planarity

$$m \leq 3n - 6$$

$$\sum_{i \in V(G)} d_i \leq 6n - 12$$

first assume in our degree sequence all vertices are of degree 6

$$\sum_{i \in V(G)} d_i = \underbrace{6n}_{\leq 6n-12} \rightarrow \text{obviously can't hold}$$

How many vertices of degree 5 or lesser do we need to hit the bound?

$$\sum_{i \in V(G)} d_i = 6(n-x) + 5x = 6n - 12$$

$$6n - x = 6n - 12$$

$$x = 12 \rightarrow \text{at least 12}$$

2 (if all other  $d_i = 6$ )